Group Identification Problems: Different Notions of Liberalism and Infinite Individuals

Federico Fioravanti
INMABB, Universidad Nacional del Sur
CONICET
Av. Alem 1253, (8000) Bahía Blanca, Argentina
e-mail:federico.fioravanti9@gmail.com,
phone: 54-291-4359890

Abstract

We study the problem of identifying members of a single group based on the opinions of the individuals in the society. The work by Kasher and Rubinstein (K-R)(1997) deals with the problem in a social choice framework. They provide axiomatic characterizations of different kinds of aggregator functions. We analyze the problem both in a finite and an infinite setting. In the former case, we focus on liberal aggregator functions, slightly modifying the axioms presented by K-R. In this way we find new uniqueness, possibility as well as impossibility results.
In the infinite case, we show that the result of the Strong Liberal CIF (introduced by K-R) still holds but not the oligarchic aggregator.
JLE:D71

1 Introduction

People, countries as well as inanimate objects are customarily classified in groups. Many times this classification is simple and obvious, as for instance organizing countries by the continent to which they belong. However, if we want to identify the members of a particular community or group countries according their degree of "eco-friendliness", the classification is far from self-evident, and the assessment of the individuals or nations involved matter for the final result. Kasher (1993) and Kasher and Rubinstein (K-R)(1997) deal with the latter version of the problem ("Who is a J?") analyzing in a Social Choice-theoretical framework, presenting the issue as a problem of defining appropriate opinion aggregation function. Each individual in a society is assumed to have an opinion about which individuals, including himself, belong or not to a group. The way to determine the identities of the individuals is through a function that take their opinions as input. K-R axiomatize these aggregator functions, which they call Collective Identity Functions (CIF) and characterize three kinds, each of which embodies a slightly different notion of "fairness". So, the "Liberal" CIF labels J any individual that deems herself J; the "Dictatorial" function is such that a single individual decides who is J and the "Oligarchic" one when this decision corresponds to a given group. In their work they provide various results about the existence (or not) and uniqueness of CIF. Since then, a lot of results have been found, by either modifying K-R's axioms (Saporiti 2012), correcting previous results (Sung and Dimitrov 2003), working with the identification of more than two groups (Cho and Ju 2016) and even dealing with the incentives of voters (Cho and Saporiti 2015).

The purpose of this paper is twofold. On one hand, in the line of the aforementioned developments, we introduce further variants of the original problem to analyze the impact of Liberalism on the properties of CIF. More precisely, we study the consequences of replacing K-R's version of Liberalism by a more classical one, an adaptation of Sen´s Liberalism (Sen 1970), and then by a variant we call Extreme Liberalism.

On the other hand, we also analyze which results of K-R are still valid when the number of voters is infinite. The idea is to examine if a similar result as Fishburn's (1970) is valid in this context.[1]

[1]Fishburn shows that Arrows Impossibility Theorem no longer holds when the number of agents is infinite.

2 Finite Case

2.1 Model

We consider a set N of individuals, with $\mid N \mid < \infty$. Each individual i has an "opinion" described by a set $J_i \subseteq N$ of individuals that i thinks belong to class J. On the other hand, if $j \notin J_i$, then i does not believe that j is in class J. By a slight abuse of language we denote by J a Collective Identity Function taking as argument a profile of opinions $(J_1, \ldots, J_N)$ and yielding a set $J(J_1, \ldots, J_N) \subseteq N$, where the $i \in J(J_1, \ldots, J_N)$ are the individuals deemed constitute class J. For simplicity, we will just identify J with $J(J_1, \ldots, J_N)$ when there is no chance of confusion.
Let us now present axioms capturing properties that are desirable for a fair CIF:

- **Monotonicity**(MON): suppose that $i \in J(J_1, \ldots, J_N)$. Let $(J'_1, \ldots, J'_N)$ be a profile identical to $(J_1, \ldots, J_N)$ except that there are individuals, i and k, so that $i \notin J_k$ and $i \in J'_k$; then $i \in J(J'_1, \ldots, J'_N)$. Analogously, if $i \notin J(J_1, \ldots, J_N)$ and if $(J'_1, \ldots, J'_N)$ is identical to $(J_1, \ldots, J_N)$, except that there is a k such that if $i \in J_k$ and $i \notin J'_k$, then $i \notin J(J'_1, \ldots, J'_N)$.

- **Independence**(I): consider two profiles $(J_1, \ldots, J_N)$ and $(J'_1, \ldots, J'_N)$ and let i and individual in N. If for every $k \neq i$, $k \in J$ if and only if $k \in J'$, and for all k (i inclusive) $i \in J_k$ if and only if $i \in J'_k$, then $i \in J$ if and only if $i \in J'$.

- **Consensus**(C): if $j \in J_i$ for all i, then $j \in J$; if $j \notin J_i$ for all i, then $j \notin J$.

- **Symmetry**(SYM): j and k are symmetric in a profile $(J_1, \ldots, J_N)$ if

  (i) $J_j - \{j, k\} = J_k - \{j, k\}$

  (ii) for all $i \in N - \{j, k\}, j \in J_i$ iff $k \in J_i$

  (iii) $j \in J_j$ iff $k \in J_k$

  (iv) $j \in J_k$ iff $k \in J_j$

  Then, $j \in J$ if and only if $k \in J$.

- **The Liberal Principle**(L): if there is an i such that $i \in J_i$, then $J \neq \emptyset$, and if there is an i such that $i \notin J_i$, then $J \neq N$.

A particular CIF introduced by K-R is the **Strong Liberal** one, defined as:

$$J = \{i \mid i \in J_i\}$$

i.e., are J is the class of all the individuals that consider themselves to be in J.
It follows that:

Theorem 1 *The Strong Liberal CIF is the only CIF that satisfies axioms (C), (SYM), (MON), (I) and (L).*

The Liberal Principle was introduced by K-R to capture the idea that if someone thinks he is in J, then *somebody* must be in J, and if somebody thinks he is not in J, then *not everyone* can be in J. We consider, instead, a variant of the concept of liberalism of Sen, that prescribes that *every* individual is *decisive* over a pair of alternatives. Formally:

- **Liberalism**(SL): for each $i \in N$, there exists a $j \in N$ such that if $j \in J_i$, then $j \in J$; and if $j \notin J_i$, then $j \notin J$.

In his work, Sen also uses weaker versions of liberalism: Minimal Liberalism and Super Minimal Liberalism.

- **Minimal Liberalism**(ML): there exists at least an $i, j \in N$, $i \neq j$, and $k, l \in N$, $k \neq l$, such that if $k \in J_i$ then $k \in J$, if $l \in J_j$ then $l \in J$, if $k \notin J_i$ then $k \notin J$ and if $l \notin J_j$ then $l \notin J$.

- **Super Minimal Liberalism**(SML): there exists at least $i, j \in N$, $i \neq j$, and $k, l \in N$, $k \neq l$, such that if $k \in J_i$ then $k \in J$ or if if $k \notin J_i$ then $k \notin J$ and if $l \in J_j$ then $l \in J$ or if $l \notin J_j$ then $l \notin J$.[2]

Sen obtained an impossibility result combining unanimity (a variant of Consensus), unrestricted domain and one of the three properties of liberalism. In our context this is no longer so. But the downside is that the uniqueness found by K-R no longer holds in the case of SML.

[2]In words: ML means that at least two agents are, each one, *decisive* over an agent while SML prescribes that at least two agents are, again each one, *semidecisive* over an agent.

Theorem 2 *The Strong Liberal CIF is not the only CIF that verifies (C), (MON), (I), (SYM) and (SML).*

Proof of Theorem 2: *Clearly, the Strong Liberal CIF satisfies these five axioms. Consider now the Unanimity CIF, that prescribes that i is in J if and only if everybody thinks that i is in J. Formally*

$$J = \{i \mid i \in J_k \ for\ all\ k \in N\}$$

This CIF satisfies the five properties, and we have that all the agents are semidecisive over any other agent, because if, for example, $i \notin J_j$, then $i \notin J$. □

Another CIF that satisfies the axioms is the "whatever" CIF, indicating that i is in J if someone thinks that i is in J:

$$J = \{i \mid i \in J_k \ for\ any\ k \in N\}$$

Again, we have that every agent is semidecisive over any other agent, since if, for example, $i \in J_j$, then $i \in J$.

When we ask the agents to be decisive, the uniqueness of the Strong Liberal CIF reappears, since it is the only CIF that satisfies all these properties:

Theorem 3 *The Strong Liberal CIF is the only CIF that verifies (C),(MON), (SYM), (I) and (SL)*

Proof of Theorem 3: *First, we see that two agents can not be decisive over the same agent. Suppose, on the contrary, that i and j are decisive over k. If in the profile it appears $k \in J_i$ and $k \notin J_j$, then the CIF will not be able to determine if k is in J or if k is not in J. Since we are working with a finite group of voters, for every CIF that satisfies (SL), any agent can be decisive over just one agent. So any CIF with this property must have the following form:*

$$J_\sigma = \{\sigma(i) \mid \sigma(i) \in J_i\}$$

where σ is a permutation of the set N of individuals.

If σ is the identity permutation we obtain the Strong Liberal CIF, satisfying the five axioms.

Now suppose that $\sigma(i) \neq i$ for some i. Then we have that J_σ satisfies (C), (MON), (I), (SL) but not (SYM). Suppose to the contrary that it satisfies (SYM). Thus, given any σ, for every possible profile $(J_1, \ldots, J_N)$, if there exists a pair j, k of symmetric individuals, either $j, k \in J_\sigma$ or $j, k \notin J_\sigma$. But a simple example shows that this not necessarily the case. Suppose that

$N = \{1,2,3\}$ and $\sigma = (23)$ (the permutation that exchanges 2 and 3, leaving 1 fixed). Consider the profile $P = (\{3\},\{1,3\},\{1\})$. It can be easily checked that agents 1 and 3 are symmetric. By definitions $J_\sigma(P) = \{3\}$, but according to (SYM) it should be $J_\sigma(P) = \{1,3\}$. This shows that (SYM) is not satisfied.$\square$

It follows that:

Corollary 1 *The Strong Liberal CIF is the only CIF that verifies (C),(MON), (SYM), (I) and (ML)*

Now we will introduce a more radical concept according to which if someone considers there is an individual in J, then J cannot be empty, and if someone thinks that there is an individual that does not belong to J, then not everybody can be in J. Formally:
Extreme Liberalism(EL) is characterized by:

(i) If there are $i, j \in N$ such that $j \in J_i$, then $J \neq \emptyset$.

(ii) if there are $i, j \in N$ such that $j \notin J_i$, then $J \neq N$.

An example of a CIF that satisfies (EL) is the following:

$$J = \begin{cases} A \subset N & if \quad J_i \neq N\ for\ all\ i \\ N & if \quad J_i = N\ for\ all\ i \\ \emptyset & if \quad J_i = \emptyset\ for\ all\ i \end{cases}$$

When we ask for a notion so extreme of liberalism, we find that the Strong Liberal CIF does not satisfy (EL). In fact, there is no CIF that can verify the conditions imposed by K-R when the liberalism axiom is (EL):

Theorem 4 *There is no CIF that verifies (C), (MON), (SYM), (I) and (EL).*

To prove this result, we will use three lemmas.

Lemma 1 *If a CIF verifies (I), then it is such that given two profiles $(J_1, \ldots, J_N)$ and $(J'_1, \ldots, J'_N)$, if $i \in J_k$ if and only if $i \in J'_k$ for every k, then $i \in J$ if and only if $i \in J'$.*

Proof of Lemma 1: *It is clear from the definition of (I) that $i \in J$ if and only if $i \in J'$.*$\square$

Lemma 2 *The only CIF that verifies (C), (SYM), (I), (MON) and part (i) of (EL) is the "whatever" CIF.*

Proof of Lemma 2: *First of all, the "whatever" CIF satisfies these five axioms. Suppose there exists a different CIF also verifying them. Consider a profile* P_1 *such that* $i \in J_j$ *for some* j *but* $i \notin J(P_1)$. *By applying (MON) several times, we arrive at a profile* P_2 *that is identical to* P_1 *with the exception that for every* $k \neq j$, $i \notin J_k$ *so that* $i \notin J(P_2)$. *Denote* $J(P_2) = M$. *Let* P_3 *be the profile such that for every* $k \in M \cup \{i\} \cup \{j\}$ *is* $J_k = \{\sigma(k)\}$ *with* $\sigma(j) = i$ *where* σ *is a permutation of* N *and* $J_k = \emptyset$ *for every* $k \notin M \cup \{i\} \cup \{j\}$ *By (C),* $J(P_3)$ *does not contain any* $k \in N - M - \{i, j\}$. *By (SYM), the CIF classifies all members of* $M \cup \{i\} \cup \{j\}$ *identically. The CIF verifies part (i) of (EL), so it is impossible that* $J(P_3) = \emptyset$ *so it must be that* $J(P3) = M \cup \{i\} \cup \{j\}$. *We then obtain a contradiction with (I), because agent* i *is treated equally in profiles* P_2 *and* P_3 *but* $i \notin J(P_2)$ *while* $i \in J(P_3)$. *Thus, there does not exist a CIF other than the "whatever" CIF satisfying the axioms.*□

Lemma 3 *the only CIF that satisfies (MON), (C), (SYM), (I) and part (ii) of (EL) is the Unanimity CIF.*

Proof of Lemma 3: *Clearly the Unanimity CIF satisfies these 5 axioms. Consider a different CIF that also verifies them. Suppose there is a profile* P_1 *such that* $i \notin J_k$ *for some* k *but* $i \in J(P_1)$. *By applying (MON) several times, we can find a profile* P_2 *identical to* P_1 *with the exception that for every* $j \neq i, i \in J_j$ *so that* $i \in J(P_2)$. *Denote* $J(P_2) = M$. *Let* P_3 *the profile such that for all* $j \in N - \{k\}$, $J_j = N$ *and* $J_k = N - \{i\}$. *By (C),* $N - \{i\} \subseteq J(P_3)$. *Because this CIF satisfies part (ii) of (EL), it is not possible that* $J(P_3) = N$, *so we have that* $J(P_3) = N - \{i\}$. *Applying Lemma 1, we obtain a contradiction with (I), because agent* i *is treated equally in profiles* P_2 *and* P_3 *but* $i \in J(P_2)$ *but* $i \notin J(P_3)$. *Then, there does not exist a CIF other than the unanimity CIF satisfying the axioms.*□
Now, we have:
Proof of Theorem 4: *A straightforward application of Lemma 2 and Lemma 3 yields that there does not exist a CIF satisfying the five axioms.*□

3 Infinite Individuals

3.1 Liberalism

We will use the same model as in the finite case, just that $N = \mathbb{N}$, the class of natural numbers. CIF is again a function that takes a profile of opinions and yields a subset of N. According to Sung and Dimitrov (2003), the properties proposed by K-R are not independent, In particular, (C) and (MON) can be derived from (SYM), (I) and (L). They use an inductive argument to prove one of the Lemmas that are necessary to obtain their main result. We adapt this proof using **transfinite induction**, while keeping the other results since their validity is independent of the cardinality of N.
The version of transfinite induction we consider is known as the **Second Principle of Transfinite Induction**: Given P a proposition such that

(i) $P(0)$ is true.

(ii) $P(\alpha)$ implies $P(\alpha + 1)$.

(iii) If α is a limit ordinal and $P(\alpha')$ is true for every $\alpha' < \alpha$, then $P(\alpha)$ is true.

Then we have:

Lemma 4 *If a CIF J satisfies (SYM), (I) and (L), then $J(S^{\mathbb{N}}) = S$ for each $S \subseteq \mathbb{N}$ where $S^{\mathbb{N}}$ is the profile where $J_i = S$ for all $i \in \mathbb{N}$.*

Proof of Lemma 4: *Sung and Dimitrov (2003) prove by finite induction on the cardinality $\mid S \mid$ of S that every CIF satisfying (SYM), (I) and (L) is such that*

$$J(S^N) = S \text{ and } J(N - S^S) = N - S \text{ for every } S \subseteq N$$

To show that this is valid in our setting just consider that their induction argument shares Steps (i) and (ii) with transfinite induction. On the other hand, step (iii) is satisfied by taking $\alpha = \mathbb{N}$.$\square$
Following the same scheme of Sung and Dimitrov, we can thus show:

Theorem 5 *The Strong Liberal CIF is the only CIF that satisfies (SYM), (I) and (L), even when the number of voters is infinite.*

3.2 Oligarchy

In this context we no longer work with individuals that have an opinion on who is in J, instead of this, each agent divides the society into two classes. Then, an aggregator function generates a partition of $\mathbb{N}$.
Formally, each $i \in \mathbb{N}$ specifies an equivalence relation on $\mathbb{N}$ denoted $\sim_i$, such that $j \sim_i k$ if i considers that j and k are in the same class. A CIF* is a function that assigns to each profile $(\sim_1, \ldots, \sim_i, \ldots)$ an equivalence relation $\sim$ over $\mathbb{N}$.
The axioms used here are the following:

- **Independence**(I^*): consider two profiles of equivalence relations, $(\sim_1, \ldots, \sim_i, \ldots)$ and $(\sim'_1, \ldots, \sim'_i, \ldots)$, such that for every i, j and k, $i \sim_k j$ if and only if $i \sim'_k j$, then $i \sim (\sim_1, \ldots, \sim_i, \ldots)$ if and only if $i \sim (\sim'_1, \ldots, \sim'_i, \ldots)$.

- **Consensus**(C^*): if $j \sim_i k$ for every $i \in \mathbb{N}$, then $i \sim j$.

An **oligarchic** CIF* is such that there exists a non-empty subset M verifying that $i \sim j$ if and only if $i \sim_k j$ for all $k \in M$. In the case that $|N| < \infty$ the following result of Barthelemy, Leclerc and Monjardet (1986) characterize oligarchic aggregators:

Theorem 6 *The only CIF* that satisfies (C^*) and (I^*) are oligarchic.*

When the number of individuals is infinite, this result does no longer hold. We have instead:

Theorem 7 *If $N = \mathbb{N}$, the oligarchic CIF* is not the only CIF* that satisfies (C^*) and (I^*).*

Proof of Theorem 7: *consider the CIF* defined in the following way:*

$$i \sim j \text{ if and only if } \{k | i \sim_k j\} = \mathbb{N} \text{ or } \mathbb{N} - \{k | i \sim_k j\} \text{ is infinite, with } \{k | i \sim_k j\} \neq \emptyset.$$

Clearly this CIF* satisfies both (C^*) and (I^*). But it is not an oligarchic CIF*.[3] *Suppose there is a proper subset of $\mathbb{N}$, M (finite or not) that conforms an "oligarchy". It cannot be decisive, because if everybody else (except a finite number of individuals) agrees the members of M, the choice of the oligarchy will not obtain.*□

[3] According to this function, i and j are deemed to belong to the same class if either all the individuals agree on that or the number of those that are against this is infinite. This makes this CIF* "weakly anti-mainstream".

4 Conclusion

In this work we deal with two issues. On one hand we consider different notions of liberalism, obtaining in some cases similar results to the ones reached by K-R while in the case of Extreme Liberalism we get an impossibility result.
On the other hand, in the infinite case we showed the power of the Strong Liberal CIF proposed by K-R, since their uniqueness result is kept. Different is the case of oligarchic CIF, where another aggregator satisfies the same desirable properties.
The dictatorial case in the infinite case, in which the range of the function cannot be the entire society nor the empty set, is matter for further research.

5 References

- Barthelemy, J.R., Leclerc, B. and Monjardet, B., 1986, "On the Use of Ordered Sets in Problems of Comparison and Consensus of Classifications", *Journal of Classification*, 3, 187-224.

- Cho, W.J. and Ju B.-G., 2016, "Multinary Group Identification", *Theoretical Economics*, forthcoming.

- Cho, W.J. and Saporiti, A., 2015, "Incentives, Fairness, and Efficiency in Group Identification", *The School of Economics Discussion Paper Series* 1117, Economics, The University of Manchester.

- Fishburn, P.C., 1970, "Arrow´s Impossibility Theorem: Concise Proof and Infinite Voters", *Journal of Economic Theory* 2(1) 103-106

- Kasher, A., 1993,"Jewish Collective Identity", in: *Jewish Identity*, edited by David Theo Goldberg and Michael Kraus, Philadelphia: Temple University Press.

- Kasher, A. and Rubinstein, A., 1997, "On the Question "Who is a J?": A Social Choice Approach", *Logique & Analyse* 160 385-395.

- Saporiti, A., 2012, "A Proof for 'Who is a J' Impossibility Theorem", *Economics Bulletin*, 32(1) 494-501.

- Sen, A., 1970, "The impossibility of a Paretian Liberal", *Journal of Political Economy* 78(1) 152-157.

- Sung, S.C. and Dimitrov, D., 2003, "On the Axiomatic Characterization of 'Who is a J?"', *Logique & Analyse* 48 101-112.